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THE HOMOTOPY TYPES OF COMPACT LIE GROUPS

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ABSTRACT

Hans Scheerer proved that if two simply connected compact Lie groups are homotopically equivalent, then the groups are isomorphic. We give a conceptually simpler proof which shows that the result depends only on the 2 and 3 primary homotopy of the Lie groups.

1. The Main Theorem

In [9] Hans Scheerer proved,

THEOREM 1. *Let L and M be compact, connected, simply connected Lie groups. Assume that the spaces of L and M are homotopically equivalent. Then L and M are isomorphic Lie groups.*

The proof given in [9] is not difficult but it gives little insight into why the result should be true. In this note we show how the theory of Finite H-spaces helps in understanding Theorem 1.

The proof of [9] and that given below both assume the known classification of simple Lie groups. Ideally one would like to give a proof which does not, but as it is not known how to recognize Lie groups among topological groups which have the homotopy types of closed smooth manifolds it is not clear how such a proof would proceed. The idea behind our proof is simple. We know that each Lie group as in Theorem 1 is isomorphic to an essentially unique finite direct product of simple Lie groups. We seek a corresponding unique cartesian product factorization theorem in homotopy and such results rarely exist. But when a simply connected Finite H-space is localized at a prime it does factorize uniquely into indecomposable H-spaces. Further at the prime 2, the localization of almost

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all the simple Lie groups are indecomposable and have distinct homotopy types. As localization preserves cartesian products, but for the "almost all" in the last sentence, the theorem follows.

2. The simple Lie groups

The symbol G will be reserved for a compact, connected simply connected Lie group.

THEOREM 2.1. *A non-trivial G is isomorphic to a finite direct product of groups taken from the following list:*

(a) $\mathrm{SU}(n+1)$, $n \geqq 1$; *the group of complex $(n+1)\times(n+1)$-matrices A satisfying $A\bar{A}^t = I$, and* $\det A = 1$,

(b) $\mathrm{Sp}(m)$, $m \geqq 2$; *the group of quaternionic $(m \times m)$-matrices B satisfying $B\bar{B}^t = I$,*

(c) $\mathrm{Spin}(q)$, $q \geqq 7$; *the group which is the double and therefore the universal covering of the group of rotations* $\mathrm{SO}(q)$,

(d) G_2, F_4, E_6, E_7 *and* E_8; *the five exceptional simply connected Lie groups.*

A discussion of this theorem can be found in Chapter XXI of [4].

The cohomology rings with Z_2-coefficients of these simple groups were calculated by the efforts of several mathematicians, particularly Borel and Araki in the 1950's and early 1960's. Convenient references are [10] for the classical groups and [11] for the exceptional groups.

THEOREM 2.2. (a) $H^*(\mathrm{SU}(n+1), Z_2) \cong \Lambda(x_3, x_5, \ldots, x_{2n+1})$, *an exterior algebra over Z_2 on generators x_{2i+1} of dimension $2i+1$.*

(b) $H^*(\mathrm{Sp}(m), Z_2) \cong \Lambda(x_3, x_7, \ldots, x_{4m-1})$.

(c) $H^*(\mathrm{Spin}(q), Z_2) \cong Z_2[x_3, x_5, \ldots, x_{2s+1}, y_{2t-1}]/I$ *where* $2s+1 < q \leqq 2s+3$ *and $t = 2^u$ if $2^u < q \leqq 2^{u+1}$. The ideal I is generated by y_{2t-1}^2 and for each i by $x_{2i+1}^{2^\alpha}$ where α is the smallest integer such that $(2i+1)2^\alpha \geqq q$.*

In (a), (b) *and* (c), *the action of the Steenrod squares on the generators x_{2j+1} is determined by formula*

$$Sq^{2i} x_{2j+1} = \binom{2j+1}{2i} x_{2i+2j+1}$$

with the usual convention that if there is no generator of dimension $2i+2j+1$, then $x_{2i+2j+1} = 0$.

(d) $H^*(G_2, Z_2) \cong Z_2[x_3, x_5]/(x_3^4, x_5^2)$,

$H^*(F_4, Z_2) \cong Z_2[x_3, x_5, x_{15}, x_{23}]/(x_3^4; x_i^2, i > 3)$,

$H^*(E_6, Z_2) \cong Z_2[x_3, x_5, x_9, x_{15}, x_{17}, x_{23}]/(x_3^4; x_i^2, i > 3)$,

$$H^*(E_7, Z_2) \cong Z_2[x_3, x_5, x_9, x_{15}, x_{17}, x_{23}, x_{27}]/(x_3^4, x_5^4, x_9^4; x_i^2, i > 9)$$
$$H^*(E_8, Z_2) \cong Z_2[x_3, x_5, x_9, x_{15}, x_{17}, x_{23}, x_{27}, x_{29}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4; x_i^2, i > 15).$$

It is always true that $Sq^2x_3 = x_5$. *For* F_4, E_6, E_7 *and* E_8, *one has* $Sq^8x_{15} = x_{23}$. *For* E_6, E_7, E_8, *one also has* $Sq^4x_5 = x_9$, $Sq^8x_9 = x_{17}$, $Sq^2x_{15} = x_{17}$. *For* E_7 *and* E_8, *one has* $Sq^4x_{23} = x_{27}$ *while for* E_8, $Sq^2x_{27} = x_{29}$.

An immediate conclusion to be drawn from Theorem 2.2 is that different simple Lie groups have distinct homotopy types. However, if one compares the cohomology of say $SU(2) \times SU(4)$ and $Sp(2) \times SU(3)$ or more significantly, $Spin(7) \times Spin(7)$ and $G_2 \times Spin(8)$ one sees that they are isomorphic as algebras over the Steenrod algebra.

3. Indecomposable spaces

An H-space X or (X, e, m) is a triple with X a topological space, e a base point in X and $m : X \times X \to X$ a continuous multiplication such that $m(e, \ ) \simeq$ Identity $\simeq m(\ , e): X \to X$. If X is an H-space and $Y \simeq X$, then Y is an H-space. Clearly G is an H-space and so is a localization at a prime p [7]. We will consider G localized at the prime 2.

A non-trivial space or homotopy type W is indecomposable if $W \simeq W_1 \times W_2$ implies that one of W_1 or W_2 is homotopically trivial.

The following theorem is of some independent interest although the result will not come as a surprise to some mathematicians.

THEOREM 3. *Let G be a non-trivial simple Lie group localized at the prime* 2.
(a) *If* $G \neq$ Spin(7) *or* Spin(8), *then it is indecomposable.*
(b) $Spin(7) \simeq G_2 \times S^7$.
(c) $Spin(8) \simeq G_2 \times S^7 \times S^7$.

There is more than one way of approaching the proof depending upon what further results one chooses to quote from the literature. We will assume certain classical results about elements of Hopf invariant one and some generalizations. Thus if X is a simply connected H-space and $H^*(X, Z_2) \cong \Lambda(x_{2i+1})$, then at the prime 2, $X \simeq S^3$ or S^7, [1] or [3]. If $H^*(X, Z_2) \cong \Lambda(x_{2i+1}, x_{2j+1})$, it is known that at the prime 2, $X \simeq S^3 \times S^3$, $S^3 \times S^7$, $S^7 \times S^7$, SU(3) or Sp(2): For our purposes, we will only need to know that $\{2i+1, 2j+1\} = \{3,3\}, \{3,5\}, \{3,7\}$ or $\{7,7\}$ [2] and [5] and, to shorten one argument, that if $\{3,5\} = \{2i+1, 2j+1\}$ then $X \simeq SU(3)$ [6]. We will also quote some standard results about principal fibre bundles of Lie groups. Except when considering fibrations, it will be assumed that all spaces are localized at the prime 2 which leaves the mod 2 cohomology rings unaffected.

PROOF OF THEOREM 3. Let G be a simple Lie group and suppose that $G \simeq X \times Y$; so X and Y are H-spaces. Now $H^3(G, Z_2) \cong Z_2$ and $H^*(G, Z_2) \cong H^*(X, Z_2) \otimes H^*(Y, Z_2)$, so we can assume that $H^3(X, Z_2) \cong Z_2$ and $H^3(Y, Z_2) \cong 0$. To prove that Y is trivial it is sufficient to show that $H^i(Y, Z_2) = 0$ for $i > 0$. The proof of Theorem 3 considers $H^*(G, Z_2)$ for simple groups case by case. It will follow easily that

$$H^*(Y, Z_2) = \Lambda(x_{2\alpha+1}, \ldots, x_{2\omega+1})$$

or is trivial and we write $t(Y) = \{2\alpha + 1, \ldots, 2\omega + 1\}$ or ϕ respectively.

(a) SU$(n+1)$, $n \geqq 1$. We check that as $x_3 \in H^3(X, Z_2)$, so does each x_{2i+1}, except possibly x_{2n+1} if $2n + 1 = 2^\alpha - 1$. For if $2i + 1 \neq 2^\beta - 1$, we can write $2i + 1 = 2^s + (2j + 1)$ where $0 \leqq 2j < 2^s - 2$ and so $Sq^{2j+2}x_{2^s-1} = x_{2i+1}$. Also if $2^s - 1 \neq 2n + 1$ and $s > 2$,

$$Sq^4 x_{2^s-3} = Sq^2 x_{2^s-1} = x_{2^s+1} \neq 0.$$

But if $x_{2j+1} \in H^*(X, Z_2)$ then so does $Sq^{2k}x_{2j+1}$ for all k and so a routine inductive argument implies that $t(Y) = \phi$ or, if $2n + 1 = 2^{\alpha-1}$, $t(Y) \subset \{2n + 1\}$. The comments on the Hopf invariant above imply that $t(Y) = \phi$ unless $2n + 1 = 7$. So it remains only to show that SU(4) is indecomposable. But if SU(4) $\simeq X \times S^7$ then $H^*(X, Z_2) \cong \Lambda(x_3, x_5)$ and so $X \simeq$ SU(3). The characteristic map of the principal fibration SU(3) $\to$ SU(4) $\to S^7$ has order 6 in π_6 (SU(3)) $\cong Z_6$ and so at the prime 2, SU(4) $\not\simeq$ SU(3) $\times S^7$. This contradiction implies the result.

(b) Sp(m), $m \geqq 2$. We show first that Sp(2) is indecomposable. The characteristic map of the principal fibration $S^3 =$ Sp(1) $\to$ Sp(2) $\to S^7$ has order 12 in $\pi_6(S^3) \cong Z_{12}$ and so in the cellular decomposition Sp(2) $= S^3 \cup e^7 \cup e^{10}$, the seven cell is attached in an essential manner mod 2. Therefore Sp(2) is indecomposable. Also as the 7-skeleton of Sp(n) for $n > 2$ is the same as that of Sp(2), the seven cell is always attached in an essential manner mod 2. Therefore $7 \not\in t(Y)$ for any n. Now using the Steenrod algebra and arguing as in (a) we deduce that $t(Y) = \phi$ for all $n \geqq 2$.

(c) Spin(q), $q \geqq 7$. Again the Steenrod squares are used as in (a) and (b), but this time there is the additional generator y_{2t-1} to consider. The three cases $2^{s-1} + 1 < q < 2^s$, $q = 2^s$ and $q = 2^s + 1$ are considered separately.

In the first case $t = 2^{s-1}$ and $t(Y) \subset \{2^s - 1\}$. Therefore $t(Y) = \phi$ unless $q = 7$. The principal fibration $G_2 \to$ Spin(7) $\to S^7$ is classified by a generator of $\pi_6(G_2) \cong Z_3$ which is trivial mod 2. Therefore Spin(7) $\simeq G_2 \times S^7$.

If $q = 2^s$, then $t = 2^{s-1}$ and $t(Y) \subset \{2^s - 1, 2^s - 1\}$. Therefore $t(Y) = \phi$ or $Y \simeq S^{2^s-1}$ or $S^{2^s-1} \times S^{2^s-1}$. In both the latter cases S^{2^s-1} is a mod 2 H-space and so $t(Y) = \phi$ unless $q = 8$. But Spin(8) is homeomorphic to $\mathrm{Spin}(7) \times S^7$ and so $\mathrm{Spin}(8) \simeq G_2 \times S^7 \times S^7$.

If $q = 2^s + 1$, then $t = 2^s$ and $t(Y) \subset \{2^s - 1, 2^{s+1} - 1\}$. Therefore $t(Y) = \phi$ unless $q = 9$. Now $t(Y) \neq \{15\}$ and $t(Y) \neq \{7, 15\}$. Also $t(Y) = \{7\}$ is impossible, although this is the least obvious fact in this proof. For if $\mathrm{Spin}(9) \simeq X \times S^7$, we have for each i, $\pi_i(\mathrm{Spin}(9)) \cong \pi_i(X) \oplus \pi_i(S^7)$. Consulting the tables given on page 25 of [9], we see that $\pi_{22}(\mathrm{Spin}(9)) = (Z_2)^2 \oplus (Z_8)^3$. But $\pi_{22}(S^7) \cong (Z_2)^3 \oplus Z_8$, which is impossible.

(d) If G is G_2, E_6, E_7 or E_8 then once more using the Steenrod algebra, we see that $t(Y) = \phi$. If $G = F_4$ then $t(Y) \subset \{15, 23\}$ which is impossible unless $t(Y) = \phi$.

This completes the proof of Theorem 3.

4. Unique factorization

We specialize to our needs Theorem 1(i) or Theorem 2 of [12].

THEOREM 4. *Let X be a non-contractible 1-connected H-space with the homotopy type of a finite complex. Then the p-localization of X at a prime p is homotopically equivalent to a finite cartesian product ΠX_i where the X_i are indecomposable spaces which are unique up to homotopy type and the ordering of the factors.*

Now let L and M be homotopically equivalent Lie groups as in Theorem 1. We express L and M as direct products of simple Lie groups of types (a), (b), (c) and (d). We localize at the prime 2 using the fact that localization preserves products. By Theorem 4 and Theorem 3 the localizations of both L and M are equivalent to identical products of spaces of types (a)′ $\mathrm{SU}(n+1)$, $n \geqq 1$, (b)′ $\mathrm{Sp}(m)$, $m \geqq 2$, (c)′ $\mathrm{Spin}(q)$, $q \geqq 9$, (d)′ G_2, F_4, E_6, E_7, E_8, and (e)′ S^7. Each factor in this decomposition other than G_2 or S^7 can arise only from the corresponding factor in the Lie group decomposition. Therefore we can write $L \cong N \times L_1$ and $M \cong N \times M_1$ where L_1 and M_1 are direct products of groups G_2, Spin(7) and Spin(8).

5. The proof of Theorem 1

One can complete the proof by noticing that if one localizes at the prime 3 and expresses G as a product of indecomposable factors, then the number of S^7

factors present equals the number of Spin(8) factors in the Lie group decomposition. A variant of this argument is as follows.

Recall that

(1) $H^*(G_2, Z_3) \cong \Lambda(x_3, x_{11})$,

(2) $H^*(\mathrm{Spin}(7), Z_{(3)}) \cong \Lambda(x_3, x_7, x_{11})$ and

(3) $H^*(\mathrm{Spin}(8), Z_3) \cong \Lambda(x_3, x_7, x_{11}, y_7)$.

In (2) and (3), $\mathscr{P}^1 x_3 = x_7$. This follows from the Steenrod module isomorphisms:

$$H^*(\mathrm{Spin}(2r+1); Z_3) \cong H^*(\mathrm{Sp}(r); Z_3),$$

$$H^*(\mathrm{Spin}(2r+2); Z_3) \cong H^*(\mathrm{Sp}(r) \times S^{2r+1}; Z_3),$$

plus our analysis of $\mathrm{Sp}(n)$ in Part 3 which shows that seven cell is attached in an essential manner mod 3, not just mod 2. It follows that the number of Spin(8) factors in any G which is isomorphic to a direct product of groups G_2, Spin(7) and Spin(8) equals the dimension of the cokernal of $\mathscr{P}^1 : H^3(G, Z_3) \to H^7(G, Z_3)$, in particular this is true for L_1 and M_1. As

$$H^*(L, Z_3) = H^*(N, Z_3) \otimes H^*(l_1, Z_3)$$

and

$$H^*(M, Z_3) \cong H^*(N, Z_3) \otimes H^*(M_1, Z_3),$$

it follows that L and M have the same number of Spin (8) factors.

Once more considering the 2-localizations of L and M, as the S^7 factors arise only from Spin(7) and Spin(8), it follows that L and M have the same number of Spin(7) factors. Therefore they have the same number of G_2 factors. Therefore $L \cong M$ as required.

Finally we note that in Theorem 1, the hypothesis that $L \simeq M$ could be weakened to $L \simeq_2 M$ and $L \simeq_3 M$. Indeed this is the best possible result as if $p \neq 2$, $\mathrm{SU}(2) \times \mathrm{SU}(4) \simeq_p \mathrm{SU}(3) \times \mathrm{Sp}(2)$ and if $p \neq 3$, $G_2 \times \mathrm{Spin}(8) \simeq_p \mathrm{Spin}(7) \times \mathrm{Spin}(7)$.

References

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20–104.
2. J. F. Adams, *H-spaces with few cells*, Topology **1** (1962), 67–72.
3. J. F. Adams and M. F. Atiyah, *K-theory and the Hopf invariant*, Q. J. Math. **17** (1966), 165–193.
4. J. Dieudonné, *Treatise on Analysis*, Vol. V, Academic Press, New York, 1977.
5. R. R. Douglas and F. Sigrist, *Sphere bundles over spheres and H-spaces*, Topology **8** (1969), 115–118.
6. P. J. Hilton and J. Roitberg, *On the classification problem for H-spaces of rank 2*, Comm. Math. Helv. **46** (1971), 506–516.

7. P. J. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, Notas de Matemática 15, North-Holland, Amsterdam, 1975.

8. M. Mimura, *The homotopy groups of Lie groups of low rank*, J. Math. Kyoto Univ. **6** (1967), 131–176.

9. H. Scheerer, *Homotopieöquivalente Kompooste Liesche Gruppen*, Topology **7** (1968), 227–232.

10. N. E. Steenrod, *Cohomology Operations*, Annals of Math. Studies 50, Princeton University Press, 1962.

11. E. Thomas, *Exceptional Lie groups and Steenrod squares*, Mich. J. Math. **11** (1964), 151–156.

12. C. Wilkerson, *Genus and cancellation*, Topology **14** (1975), 29–36.